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Non-polynomial conservation law densities generated by the symmetry operators in some hydrodynamical models

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Abstract
A new extra series of conserved densities for the polytropic gas model and nonlinear elasticity equation is obtained without any reference to the recursion operator or to the Lax operator formalism. Our method is based on the utilization of the symmetry operators and allows us to obtain the densities of arbitrary homogeneity dimensions. The non-polynomial densities with logarithmic behaviour are presented as an example. Special attention is paid to the singular case ($\gamma = 1$) for which we have found new non-homogeneous solutions expressed in terms of the elementary functions.

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1. Introduction

The conservation laws are the most important object in classical mechanics as well as in field theory. There are many different methods of constructing these laws. The most popular, especially used in soliton theory and in hydrodynamics, utilizes the so-called recursion operator or Lax operator formalism [1]. On the other hand, it appears that, for the nonlinear Schrödinger equation, it is possible to find one series of conserved Hamiltonians using the recursion operator only. However, for the shallow water equation, which is the dispersionless limit of the nonlinear Schrödinger equation, there is an additional series of conserved densities which is impossible to obtain using the recursion operator (see, for example, [2]).

In this paper we would like to show that it is possible to construct a new extra series of conserved densities for the polytropic gas model and nonlinear elasticity equation [3, 4] avoiding using the recursion operator or Lax formalism. More precisely, we generate many non-equivalent Hamiltonians of the given dimensions, using the symmetry operator. Our Hamiltonians are non-polynomial expressions which contain logarithmic functions. Independently, we consider the singular case, for the polytropic gas system for which $\gamma = 1$.

For this system we have constructed a new series of non-homogeneous solutions expressed in terms of the elementary functions.

The paper is organized as follows. In section 2 we describe the basic properties of the polytropic gas system which we use in the next sections. In section 3 we describe our symmetry approach where we utilize the shift, scaling and projective operators in order to generate the conserved densities. In section 4 we present explicitly a new series of the conserved densities for different, physically interesting, models of the polytropic gas system with $\gamma = 2, 3, 4, 5, \frac{5}{3}, \frac{7}{5}, -1$. In section 5 we adopt our formalism to the degenerated case ($\gamma = 1$) which is obtained from the contraction of the previous case. We show that, in this case, the conserved densities are connected with the Bessel equation.

2. Hydrodynamical systems

The theory of hydrodynamical-type systems of the nonlinear equations [5]

$$u_t^i = \sum_{j=1}^{N} \upsilon_j^i(\mathbf{u}) u_x^j \qquad i, j = 1, 2, \ldots, N \tag{1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ and υ_j^i are some functions, integrable by the generalized hodograph method [6], is closely related to the overdetermined systems of first-order partial differential linear equations. The conservation laws are such that

$$\frac{\partial h}{\partial t} = \frac{\partial g}{\partial x} \tag{2}$$

where $h(\mathbf{u})$ is density and $g(\mathbf{u})$ is flux. Then, densities satisfy the following linear system of the first order

$$\partial_k(\partial_i h) = \Gamma_{ik}^i(\partial_i h) + \Gamma_{ik}^k(\partial_k h) \qquad i \neq k \tag{3}$$

where

$$\Gamma_{ik}^i \equiv \frac{\partial_k \mu^i}{\mu^k - \mu^i} \qquad i \neq k \qquad \partial_k \equiv \partial/\partial r^k \tag{4}$$

and $r^k(\mathbf{u})$ are the so-called Riemann invariants. The hydrodynamic-type system (1) for $N = 2$ can be rewritten in the diagonal form

$$r_t^i = \mu^i(\mathbf{r}) r_x^i \tag{5}$$

and there is no summation on the repeated indices. However, for an arbitrary $N > 2$ the diagonalizability does not hold in general [13]. Thus, equation (3) is a linear system of first-order partial differential equations with variable coefficients. The general solution of such a system is determined up to N arbitrary functions of a single variable.

There are many particular cases of the system (3) for which a general solution is expressed in explicit and in compact form (see [7]). If we cannot find a general solution of such a system, then an alternative way to solve the Cauchy or Goursat problems is to create an infinite number of particular solutions [8].

It appears that the conserved densities can be used in the construction of particular solutions. Indeed, let us consider the polytropic gas

$$\eta_t = \partial_x(u\eta) \qquad u_t = \partial_x \left[\frac{u^2}{2} + \frac{\eta^{\gamma-1}}{\gamma - 1} \right] \tag{6}$$

and nonlinear elasticity equations

$$\eta_y = u_x \qquad u_y = \partial_x \left[\frac{\eta^{\gamma-2}}{\gamma - 2} \right] \tag{7}$$

which are commuting flows to each other.

The solution of the first system obtained by the hodograph method is

$$t = h - \eta\frac{\partial h}{\partial \eta} \qquad x = -\frac{\partial h}{\partial \eta} - \frac{u}{\eta}\frac{\partial h}{\partial u} \tag{8}$$

while for the second the solution is

$$x = \frac{\partial h}{\partial \eta} \qquad y = \frac{\partial h}{\partial u} \tag{9}$$

where h is some solution of the Tricomi-like equation

$$h_{uu} = \eta^{3-\gamma} h_{\eta\eta}. \tag{10}$$

The previous equation is nothing but the equation on the conservation law densities for both systems. Equations (8) and (9) realize the general or particular solution for both systems if h is the general or particular solution of the equation (10), respectively.

In [3] and [4] two infinite series of quasi-linear conservation laws were constructed for the polytropic gas and for the nonlinear elasticity equations, respectively. Quasi-linear means that these conservation law densities are polynomials with respect to u, η and η^γ where γ is an arbitrary polytropic constant.

However, as we see in the next section, these densities do not exhaust the set of all possible conserved densities.

3. Symmetry operator approach for $\gamma \neq 1$

Let us first consider the more general form of the polytropic gas system

$$\eta_t = \partial_x(u\eta) \qquad u_t = \partial_x\left[\frac{u^2}{2} + \eta f''(\eta) - f'(\eta)\right] \tag{11}$$

and the nonlinear elasticity equation

$$\eta_y = u_x \qquad u_y = \partial_x f''(\eta) \tag{12}$$

where f is an arbitrary function. Our generalized systems constitute Hamiltonian equations with the following local structure

$$\eta_t = \partial_x\frac{\delta H}{\delta u} \qquad u_t = \partial_x\frac{\delta H}{\delta \eta} \tag{13}$$

where

$$H = \int\left[\frac{1}{2}u^2\eta + \eta f'(\eta) - 2f(\eta)\right]\mathrm{d}x \tag{14}$$

for the generalized polytropic gas system (11) while for the generalized nonlinear elasticity equation (12) we have

$$\eta_y = \partial_x\frac{\delta \tilde{H}}{\delta u} \qquad u_y = \partial_x\frac{\delta \tilde{H}}{\delta \eta} \tag{15}$$

where

$$\tilde{H} = \int\left[\frac{u^2}{2} + f'(\eta)\right]\mathrm{d}x. \tag{16}$$

In order to obtain the conserved densities we try to eliminate the differentials dp and dq from the corresponding conservation laws for generalized polytropic gas and nonlinear elasticity systems, respectively. Differentiating the corresponding conservation laws

$$h_t = p_x \qquad h_t = q_x. \tag{17}$$

and using equations (11) and (12) by straightforward calculation we have

$$\begin{aligned} \mathrm{d}p &= [uh_u + \eta h_\eta]\,\mathrm{d}u + [\eta f'''(\eta)h_u + uh_\eta]\,\mathrm{d}\eta \\ \mathrm{d}q &= h_\eta\,\mathrm{d}u + f'''(\eta)h_u\,\mathrm{d}\eta. \end{aligned} \tag{18}$$

Then, the compatibility conditions $((q_u)_\eta = (q_\eta)_u, (p_u)_\eta = (p_\eta)_u)$ lead to the Tricomi-like equation

$$h_{\eta\eta} = f'''(\eta)h_{uu}. \tag{19}$$

Since this equation is compatible with the shift symmetry operator (see [10])

$$\delta = \partial/\partial u \tag{20}$$

we can search solutions in the form

$$h_u = \lambda h \tag{21}$$

where λ is an arbitrary parameter. This is the eigenvalue problem for the shift symmetry operator. Then the Tricomi-like equation is transformed to the linear ordinary differential equation

$$h_{\eta\eta} = \lambda^2 f'''(\eta)h. \tag{22}$$

However, this equation cannot be solved explicitly for an arbitrary f.

Let us observe that the shift symmetry operator transforms one solution of the Tricomi-like equation on to another one

$$\partial_u h_{n+1} = h_n. \tag{23}$$

This means that all conservation law densities h_k and corresponding fluxes p_k, q_k can be written in the quadratures recursively (see for instance [2, 8, 10]):

$$\mathrm{d}h_{k+1} = h_k\,\mathrm{d}u + q_k\,\mathrm{d}\eta \qquad \mathrm{d}q_{k+1} = q_k\,\mathrm{d}u + f'''(\eta)h_k\,\mathrm{d}\eta \tag{24}$$

$$\mathrm{d}p_{k+1} = [uh_k + \eta q_k]\,\mathrm{d}u + [\eta f'''(\eta)h_k + uq_k]\,\mathrm{d}\eta. \tag{25}$$

We can start this by considering some simple conservation laws such as, for example, $h_0^{(1)} = \eta, q_0^{(1)} = u$ or $h_0^{(2)} = u, q_0^{(2)} = f''(\eta)$. Interestingly, all known examples of the conserved densities could be obtained in this way (see, for example, [2, 3, 9, 12]).

However, we demonstrate different possibilities which give us a new series of the conserved densities. Our main idea is based on the following observation.

The Tricomi-like equation (10) is compatible with three local symmetry operators:

(1) shift operator

$$\delta = \frac{\partial}{\partial u} \tag{26}$$

(2) scaling operator

$$R = u\frac{\partial}{\partial u} + \frac{2}{\gamma - 1}\eta\frac{\partial}{\partial \eta} \tag{27}$$

(3) projective operator

$$S = \left[\frac{\gamma - 1}{4}u^2 + (\gamma - 1)^{-1}\eta^{\gamma-1}\right]\frac{\partial}{\partial u} + u\eta\frac{\partial}{\partial \eta} + \frac{\gamma - 3}{4}u. \tag{28}$$

These operators, with the identity operator, constitute closed Lie algebra, with the following commutation relations

$$[\delta, S] = \frac{2}{\gamma - 1} R + \frac{\gamma - 3}{4} \qquad [\delta, R] = \delta \qquad [R, S] = S. \tag{29}$$

They act on homogeneous conservation law densities as follows

$$\delta h_{k+1} = h_k \qquad R h_k = c_k h_k \qquad S h_k = h_{k+1} \tag{30}$$

where c_k are degrees of homogeneity. Thus, by combining these symmetry operators we can describe all quasi-linear conservation law densities.

Interestingly, if we rewrite the Tricomi-like equation using the Riemann invariants we obtain the famous Euler–Darboux–Poisson equation

$$\frac{\partial h}{\partial r^1 \partial r^2} = \frac{\varepsilon}{r^1 - r^2} \left[\frac{\partial h}{\partial r^1} - \frac{\partial h}{\partial r^2} \right] \tag{31}$$

where

$$\varepsilon = \frac{3 - \gamma}{2(1 - \gamma)} \qquad r^1 = u + \frac{2}{\gamma - 1} \eta^{\frac{\gamma - 1}{2}} \qquad r^2 = u - \frac{2}{\gamma - 1} \eta^{\frac{\gamma - 1}{2}}. \tag{32}$$

Now the symmetry operators are

$$\delta = \frac{\partial}{\partial r^1} + \frac{\partial}{\partial r^2} \qquad R = r^1 \frac{\partial}{\partial r^1} + r^2 \frac{\partial}{\partial r^2} \tag{33}$$

$$S = (r^1)^2 \frac{\partial}{\partial r^1} + (r^2)^2 \frac{\partial}{\partial r^2} + \varepsilon (r^1 + r^2). \tag{34}$$

The 'zero solutions' of the shift operator δ

$$\delta h_0 = 0 \tag{35}$$

can be found immediately

$$h_0^{(1)} = 1 \qquad h_0^{(2)} = \eta \tag{36}$$

while for the projective operator we can obtain

$$h_1^{(1)} = u \qquad h_1^{(2)} = u\eta \tag{37}$$

etc.

The 'zero solutions' of the projective operator are easy to obtain also from the Riemann invariant (r^1, r^2) form

$$h = (r^1 r^2)^{-\varepsilon}. \tag{38}$$

If we shift the Riemann invariants, in this formula, on arbitrary parameter λ we obtain the generating function on the conservation law densities. When $\lambda \to \infty$ we can obtain known densities.

In order to obtain new densities we demonstrate quite different possibilities. Let us consider the following recursive chain

$$R h_k^{(0)} = c_k h_k^{(0)} \qquad R h_k^{(1)} = c_k \big[h_k^{(1)} + h_k^{(0)} \big] \qquad R h_k^{(2)} = c_k \big[h_k^{(2)} + h_k^{(1)} \big] \tag{39}$$

etc, where $h_k^{(0)}$ are the quasi-linear conserved densities.

Now $h_k^{(i)}, i = 1, 2, \ldots$ are new conserved densities which satisfy the Tricomi-like equation (10). Notice that the second formula in equation (30) is not in contradiction to the previous chain because formula (30) is valid for homogeneous conservation laws only.

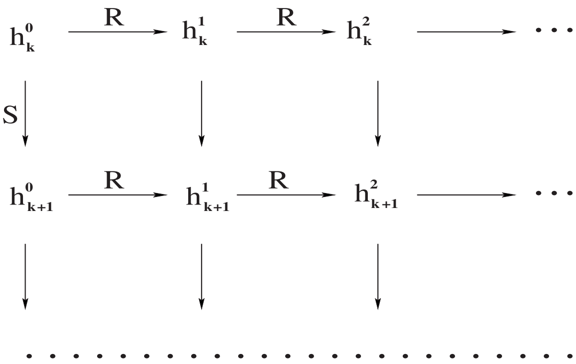

Figure 1. Generation of the nonpolynomial densities.

Now let us consider the shallow water equation, e.g. the polytropic gas system with $\gamma = 2$. Let us choose the first homogeneous solutions of equation (30)

$$h_1^{(0)} = u \qquad h_2^{(0)} = \eta \qquad h_3^{(0)} = u\eta \qquad h_4^{(0)} = u^2\eta + \eta^2 \tag{40}$$

etc. As a result we find that

$$\begin{aligned} h_1^{(1)} &= 2\sqrt{u^2 - 4\eta}(\ln\left(\sqrt{u^2 - 4\eta} + u\right) - \ln 2) + \left(u - \sqrt{u^2 - 4\eta}\right)\ln\eta \\ h_2^{(1)} &= \frac{u^2}{2} + \eta\ln\eta \\ h_3^{(1)} &= \frac{1}{4}(u^3 + 6u\eta\ln\eta) \\ h_4^{(1)} &= \frac{1}{6}(u^4 + 12u^2\eta\ln\eta + 12\eta^2\ln\eta - 18\eta^2) \end{aligned} \tag{41}$$

are conserved densities also. Now, let us apply this procedure to $h_2^{(1)}$ and $h_3^{(1)}$

$$\begin{aligned} h_2^{(2)} = \tfrac{1}{18}(&-36\eta\ln^2(\sqrt{z} + u) + 18\ln(\sqrt{z} + u)(2\eta u\ln(4\eta) - 2\eta + u\sqrt{z}) \\ &+ 9u\sqrt{z}(1 - \ln(4\eta)) - 36\eta\ln\eta\ln 2 + (9u^2 + 18\eta)\ln\eta - 18u^2 + 32\eta) \\ h_3^{(2)} = \tfrac{1}{12}(&-36\eta u\ln^2(\sqrt{z} + u) + \ln(\sqrt{z} + u)(36\eta u\ln(4\eta) - 6\eta u \\ &+ 6\sqrt{z}(u^2 + 8\eta)) - 36\eta u\ln(\eta)\ln 2 + 3\ln\eta(u^3 + 6\eta u - \sqrt{z}(u^2 + 8\eta)) \\ &+ 3\sqrt{z}(1 - 2\ln 2)(u^2 + 8\eta) - 16\eta u - 8u^3) \end{aligned} \tag{42}$$

where $z = u^2 - 4\eta$.

We have obtained rather complicated formulae which contain the logarithm in the second power. Now we can continue this procedure obtaining complicated formulae which contain the logarithm in higher powers. Furthermore, we can combine our method with this which generates quasi-linear densities, obtaining figure 1.

Figure 1 is obviously closed. In order to see this, let us start consideration from some $h_k^{(n)}$. Applying at first projective and at the next step scaling transformations, we obtain

$$RSh_k^{(n+1)} = c_{k+1}\left(Sh_k^{(n+1)} + Sh_k^{(n)}\right). \tag{43}$$

Finally, applying these operators in the reverse order we obtain

$$SRh_k^{(n+1)} = c_k\left(Sh_k^{(n+1)} + Sh_k^{(n)}\right). \tag{44}$$

In this way, our densities are determined up to an arbitrary multiplicative constant.

4. Non-polynomial densities for an arbitrary $\gamma \neq 1$ in a polytropic gas system

For an arbitrary γ, our generating equations become

$$Rh_k^{(n)} = c_k\big(h_k^{(n)} + h_k^{(n-1)}\big). \tag{45}$$

Now we have to solve the Tricomi-like equation on $h_k^{(n)}$, which is however rather hard to do. We restrict ourselves, for this reason, to two cases where $h_1^{(0)} = u$ and $h_2^{(0)} = \eta$.

For the first case, we have $c_1 = 1$. By direct verification we can check that

$$h_1^{(1)} = uh(\tau) = u \ln u \tag{46}$$

where $\tau = u^2\eta^{1-\gamma}$ satisfies equation (45). The Tricomi equation reduces in this case to

$$\tau^2(4 - (\gamma - 1)^2)\frac{\partial^2 h}{\partial^2 \tau} + (6 - \gamma(\gamma - 1)\tau)\frac{\partial h}{\partial \tau} + 1 = 0. \tag{47}$$

For the second case, we have $c_2 = \frac{2}{\gamma - 1}$. Similarly to the first case, by direct verification, we can check that

$$h_2^{(1)} = \eta h(\tau) + \eta \ln \eta \tag{48}$$

satisfies equation (45). Substituting $h_2^{(1)}$ to the Tricomi-like equation (10) we obtain the following equation on the function $h(\tau)$

$$\tau(4 - \tau(\gamma - 1))\frac{\partial^2 h}{\partial^2 \tau} - (2 - (\gamma - 1)(\gamma - 2)\tau)\frac{\partial h}{\partial \tau} + 1 = 0. \tag{49}$$

In both cases, it is possible to obtain closed formulae for particular values of the parameter γ. However, we can simplify our consideration using different coordinates. For example, choosing the coordinates r and p as (see [3])

$$\eta = rp \qquad u = \frac{1}{\gamma - 1}(r^{\gamma-1} + p^{\gamma-1}) \tag{50}$$

we can rewrite the scaling symmetry operator and Tricomi-like equation as

$$R = \frac{1}{\gamma - 1}\left(r\frac{\partial}{\partial r} + p\frac{\partial}{\partial p}\right) \qquad p^{3-\gamma}\frac{\partial^2 h}{\partial^2 p} = r^{3-\gamma}\frac{\partial^2 h}{\partial^2 r}. \tag{51}$$

Taking into account $h_1^{(0)} = r$, we have obtained the following solutions on the $h_1^{(1)}$ densities

$$\begin{aligned}
\gamma = 2 \qquad & h_1^{(1)} = -(r - p)\ln(r - p) + p\ln(p) \\
\gamma = 3 \qquad & h_1^{(1)} = (r + p)\ln(r + p) - (r - p)\ln(r - p) \\
\gamma = 4 \qquad & h_1^{(1)} = -(r - p)\ln(r - p) + (r + 2p)\ln(p^2 + pr + r^2) + \sqrt{3}p \arctan\frac{p + 2r}{\sqrt{3}p} \\
\gamma = 5 \qquad & h_1^{(1)} = (p + r)\ln(p + r) + (r - p)\ln(r - p) + r\ln(p^2 + r^2) + 2p\arctan\frac{r}{p}.
\end{aligned} \tag{52}$$

Finally, let us present the solutions for $\gamma = \frac{5}{3}, \frac{7}{5}$ and $\gamma = -1$ which are interesting from the physical point of view because these describe the dynamics of one-atomic gas and two-atomic gas [11] and two-dimensional nonlinear Born–Infeld electrodynamics [12], respectively:

$$
\begin{aligned}
\gamma = \frac{5}{3} \qquad & h_1^{(1)} = \frac{3}{4} p \operatorname{arctanh} \tau^{\frac{1}{3}} + \frac{r}{2} \ln\left(3\tau^{\frac{2}{3}} - 3\tau^{\frac{4}{3}} + \tau^2 - 1\right) - r\ln\tau - 3r\tau^{-\frac{2}{3}} \\
\gamma = \frac{7}{5} \qquad & h_1^{(1)} = \frac{5}{4} p \operatorname{arctanh} \tau^{\frac{1}{5}} + \frac{r}{2} \ln(-10\tau^{\frac{4}{5}} - 5\tau^{\frac{8}{5}} + 5\tau^{\frac{2}{5}} + \tau^{\frac{6}{5}} + \tau^2 - 1) \\
& \quad - r\ln\tau - \frac{5}{3} r\tau^{-\frac{2}{5}} - 5r\tau^{-\frac{4}{5}} + r\ln r \\
\gamma = -1 \qquad & h_1^{(1)} = \frac{1}{2}\left(r\ln\frac{p^2 r^2}{r^2 - p^2} + p\ln\frac{r-p}{r+p}\right).
\end{aligned}
\tag{53}
$$

5. Symmetry operator approach for $\gamma = 1$ and corresponding non-polynomial densities

For this case, the polytropic gas and the nonlinear elasticity systems are

$$\eta_t = \partial_x(u\eta) \qquad u_t = \partial_x\left[\frac{u^2}{2} + \ln\eta\right] \tag{54}$$

$$\eta_y = u_x \qquad u_y = \partial_x\left(-\frac{1}{\eta}\right) \tag{55}$$

respectively. For this case, the Euler–Darboux–Poisson equation (31) degenerates because $\varepsilon \to \infty$. Similarly the scaling operator R (see equation (27)) also becomes degenerated.

In order to solve this problem we notice that the analogue of the symmetry algebra (28) can be obtained considering the contraction with respect to $\gamma = 1$. If we rescale $\to \frac{\gamma-1}{2}R$ and compute the limit when $\gamma \to 1$ we obtain

$$\delta = \partial_u \qquad R = \eta\partial_\eta \qquad S = \ln\eta\,\partial_u + u\eta\partial_\eta - u/2. \tag{56}$$

Interestingly now, in the contraction limit, the homogeneity property (30) does not hold. This means that the Tricomi-like equation

$$h_{uu} = \eta^2 h_{\eta\eta} \tag{57}$$

has no homogeneous solutions.

However, the quasi-homogeneous solutions could be obtained ([9]). By quasi-homogeneous we mean homogeneous with respect to the one variable u or $\ln\eta$.

For our further applications, we describe another possibility. We have constructed the generating function for the conserved densities. To end this, we consider the eigenvalue problem for each symmetry operator.

1. For the scaling operator R (see equation (56)) the eigenvalue problem

$$h_\xi = \lambda h \tag{58}$$

where $\partial_\xi \equiv \eta\partial_\eta$, reduces the Tricomi-like equation to the linear ordinary differential equation of the second order

$$h_{uu} = \lambda(\lambda - 1)h. \tag{59}$$

These equations can be easily integrated and we obtain

$$h = \exp\left[\sqrt{\lambda(\lambda-1)}u + \lambda\xi\right]. \tag{60}$$

2. For the shift operator δ (see equation (56)) the eigenvalue problem

$$h_u = \tilde{\lambda} h \tag{61}$$

reduces the Tricomi-like equation to the linear ordinary differential equation of the second order

$$h_{\xi\xi} - h_\xi = \tilde{\lambda}^2 h. \tag{62}$$

These equations are also easy to integrate and the solutions have the same form as (60) in which we should replace $\lambda^2 \to \tilde{\lambda}(\tilde{\lambda} - 1)$.

If we expand function h near $\lambda \to 0$ or $\lambda \to 1$, then we can obtain two well-known infinite series of conservation law densities [9].

3. Projective operator S (see (56)). If we substitute

$$h = q \exp[\xi/2] \qquad u = 2s \cosh\theta \qquad \xi = 2s \sinh\theta \tag{63}$$

where s and θ are new functions, then the eigenvalue problem looks like

$$q_\theta = \lambda q. \tag{64}$$

Now the generating function is

$$q = \psi(s) \exp[\lambda\theta] \tag{65}$$

where the function ψ satisfies the Bessel equation

$$\psi'' + \frac{1}{s}\psi' + \left[1 - \frac{\lambda^2}{s^2}\right]\psi = 0. \tag{66}$$

Let us present the most simple example of the conserved densities obtained in this way, where it is possible to describe the Bessel function by the elementary functions for which $\lambda = \frac{1}{2}$

$$h = \sqrt{\frac{\eta}{u - \ln\eta}} \cos\left(\frac{\sqrt{u^2 - \ln^2\eta}}{2}\right). \tag{67}$$

Using the recursion relations, known in the theory of Bessel functions, it is possible to generate infinite sets of conserved densities written in terms of elementary functions without any reference to the recursion operator, which appears in the Hamiltonian approach to the polytropic gas systems. Finally we can use the shift or scaling symmetry operators and generate some conserved densities, as we have done in the previous sections.

6. Conclusion

In this paper we have constructed a new extra series of conserved densities for the polytropic gas model and nonlinear elasticity equation avoiding the use of the recursion operator or Lax formalism. Our Hamiltonians appeared as non-polynomial expressions which contain logarithmic functions. If we continue our procedure to these logarithmic densities in the next step, we obtain expressions with the logarithmic function in an arbitrary power. We considered the singular case of the polytropic gas system also, for which we found the non-homogeneous solutions expressed in terms of Bessel functions. If we apply the point transformation to the symmetry operators then they can change the role. This means that the scaling and shift operators are equivalent to each other, which it easy to see in the Riemann invariants (33) or (58) and (61). We have presented this method for two hydrodynamical systems only, but this method can also be adopted for more complicated equations.

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